TO SHOW THAT A DIRECT PRODUCT OF TWO BASIS SETS FOR REPRESENTATIONS IS ALSO THE BASIS FOR A REPRESENTATION

1. Must show that $\hat{\ }(X_iY_k)$ is a linear combination of the $\{X_jY_\ell\text{'s}\}$ which are the direct product basis, given $\{X_j\}$, $\{Y_\ell\}$ are basis sets for representations of the group.

$\underline{\underline{\Gamma}}^X(\hat{\ })$ is the matrix for the operator $\hat{\ }$ in a basis set $\{X_j\}$.

$$\hat{\ }X_i = \sum_{j=1}^{m}(\Gamma^X(\hat{\ }))_{ij}X_j \quad \begin{pmatrix}\{X_j\} \text{ is basis}\\ \text{of size } m\end{pmatrix}$$

$$\hat{\ }Y_k = \sum_{\ell=1}^{n}(\Gamma^Y(\hat{\ }))_{k\ell}Y_\ell \quad \begin{pmatrix}\{Y_\ell\} \text{ is basis}\\ \text{of size } n\end{pmatrix}$$

$$\hat{\ }X_iY_k = \hat{\ }(X_i)\hat{\ }Y_k \quad \begin{matrix}\text{since } \hat{\ }\text{'s act}\\ \text{independently on } \{X\}\{Y\}\end{matrix}$$

$$\hat{\ }X_iY_k = \sum_{j=1}^{m}\sum_{\ell=1}^{n}[\Gamma^X(\hat{\ })]_{ij}[\Gamma^Y(\hat{\ })]_{k\ell}X_jY_\ell$$

let $\quad [\Gamma^X(\hat{\ })]_{ij}[\Gamma^Y(\hat{\ })]_{k\ell} = [\Gamma^Z(\hat{\ })]_{ik,j\ell} = Z_{ik,j\ell}(\hat{\ })$

$$\hat{\ }X_iY_k = \sum_{j\ell=1}^{m\times n}[\Gamma^Z(\hat{\ })]_{ik,j\ell}X_jY_\ell$$

Thus the $Z_{ik,j\ell}(\hat{\ }) = [\Gamma^Z(\hat{\ })]_{ik,j\ell}$ are elements of a (mn × mn) matrix describing how the $\hat{\ }$'s take one of the direct product basis functions (X_iY_k) into a linear combination of $\{X_jY_\ell\text{'s}\}$.

2. To show that $\underline{\underline{\Gamma}}^Z(\hat{\ })$ form a representation:

given

$$\hat{\ }_A\hat{\ }_B = \hat{\ }_C \qquad \underline{\underline{\Gamma}}^X(\hat{\ }_A)\underline{\underline{\Gamma}}^X(\hat{\ }_B) = \underline{\underline{\Gamma}}^X(\hat{\ }_C)$$

$$\underline{\underline{\Gamma}}^Y(\hat{\ }_A)\underline{\underline{\Gamma}}^Y(\hat{\ }_B) = \underline{\underline{\Gamma}}^Y(\hat{\ }_C)$$

must show $\quad \underline{\underline{\Gamma}}^Z(\hat{\ }_A)\underline{\underline{\Gamma}}^Z(\hat{\ }_B) = \underline{\underline{\Gamma}}^Z(\hat{\ }_C)$

$$[\Gamma^Z(\hat{\ }_A)\Gamma^Z(\hat{\ }_B)]_{ik,j\ell} = \sum_{st=1}^{mn}[\Gamma^Z(\hat{\ }_A)]_{ik,st}[\Gamma^Z(\hat{\ }_B)]_{st,j\ell}$$

$$= \sum_{st=1}^{mn} [\Gamma^X(\hat{\ }_A)]_{is}[\Gamma^X(\hat{\ }_A)]_{kt}[\Gamma^X(\hat{\ }_B)]_{sj}[\Gamma^Y(\hat{\ }_B)]_{t\ell}$$

$$= \sum_{s=1}^{m} [\Gamma^X(\hat{\ }_A)]_{is}[\Gamma^X(\hat{\ }_B)]_{sj} \sum_{t=1}^{n} [\Gamma^Y(\hat{\ }_A)]_{kt}[\Gamma^Y(\hat{\ }_B)]_{t\ell}$$

$$[\underline{\underline{\Gamma}}^Z(\hat{\ }_A)\underline{\underline{\Gamma}}^Z(\hat{\ }_B)]_{ik,j\ell} = [\Gamma^X(\hat{\ }_C)]_{ij}[\Gamma^Y(\hat{\ }_C)]_{k\ell}$$

$$= [\Gamma^Z(\hat{\ }_C)]_{ik,j\ell}$$

Q.E.D.

3. To show that $\chi^Z(\hat{\ }) = \chi^X(\hat{\ })\chi^Y(\hat{\ })$

$$\chi^Z(\hat{\ }) = \sum_{ik=1}^{mn} [\Gamma^Z(\hat{\ })_{ik,ik}$$

$$= \sum_{ik=1}^{mn} [\Gamma^X(\hat{\ })]_{ii}[\Gamma^Y(\hat{\ })]_{kk}$$

$$= \sum_{i=1}^{m} [\Gamma^X(\hat{\ })]_{ii} \sum_{k=1}^{n} [\Gamma^Y(\hat{\ })_{kk}$$

$$\chi^Z(\hat{\ }) = \chi^X(\hat{\ })\chi^Y(\hat{\ })$$